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Degenerate toroidal magnetohydrodynamic equilibria and minimum B

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It is shown that there is a unique configuration of toroidal magnetic surfaces which has the property that it corresponds to more than one plasma equilibrium and that this is the configuration of isodynamic equilibria. These equilibria include toroidal minimum-B equilibria and the distinction between these unstable systems and the stable minimum-B mirror systems is discussed.

I. INTRODUCTION

Axisymmetric toroidal magnetohydrodynamic (MHD) equilibria are usually specified in terms of two functions such as the pressure profile $p(\psi)$ and poloidal current $f(\psi)$ [or something equivalent, such as the safety factor $q(\psi)$] together with appropriate boundary conditions. An interesting, and potentially important, question is whether an equilibrium is uniquely determined by other data. Recently, Christiansen and Taylor[1] showed that the shape of the magnetic surfaces alone can uniquely determine an axisymmetric equilibrium and described a method for constructing the current profile from the shape of the surfaces in both toroidal and linear systems. In the linear case, the construction obviously fails when the magnetic surfaces in question are concentric circular cylinders, since an infinity of different pressure and current profiles leads to equilibria with circular magnetic surfaces. In this sense circular surfaces are degenerate. This raises the question of whether there are configurations of *toroidal* magnetic surfaces that are similarly degenerate and correspond to more than one pressure or current profile. The magnetic surfaces of the isodynamic equilibria[2] (in which magnetic surfaces coincide with constant-$|B|$ contours) have this property, and we will show that they are the only such surfaces.

In Sec. II the concept of degenerate surfaces is introduced and in Sec. III we contruct these surfaces using a novel method of solution of the Grad–Shafranov equation in which each surface is calculated individually. Some properties of the corresponding equilibria are reviewed in Sec. IV. We note that they include examples of toroidal minimum-B systems which, like all isodynamic equilibria, are unstable.[2] The connection between these unstable minimum-B systems and the unconditionally stable minimum-B mirror systems is explored in Sec. V.

II. DEGENERATE EQUILIBRIA

In a cylindrical coordinate system R,ϕ,Z, the magnetic field can be written

$$\mathbf{B}=\frac{\mathbf{n}\times\nabla\psi}{R}+\frac{f(\psi)}{R}\mathbf{n}, \tag{1}$$

and in equilibrium $\psi(R,Z)$ satisfies the equation

$$\begin{aligned}\Delta^{+}(\psi)&\equiv R^{2}\nabla\cdot[(1/R^{2})\nabla\psi]\\ &=-\mu_0 R^{2}p'(\psi)-ff'(\psi),\end{aligned} \tag{2}$$

where $p(\psi)$ is the plasma pressure. If a different plasma equilibrium, with pressure $p^*(F)$ and current $f^*(F)$, has identical flux surfaces there must be a functional relation $F=F(\psi)$ between ψ and F so that

$$\Delta^{+}F=\frac{dF}{d\psi}\Delta^{+}\psi+\frac{d^{2}F}{d\psi^{2}}|\nabla\psi|^{2}. \tag{3}$$

But if F is indeed an equilibrium, the right-hand side of this expression must have the same form as the rhs of Eq. (2). This is the case if, and only if, $|\nabla\psi|^2$ can be expressed in the form

$$|\nabla\psi|^{2}=\alpha(\psi)+\beta(\psi)R^{2}. \tag{4}$$

Hence, if any solution of Eq. (2) can be found that also satisfies (4), then there will be a whole family of equilibria, with different pressure and current profiles, which have exactly the same flux surfaces. Note that the total magnetic field for these degenerate equilibria is

$$B^{2}=\frac{(|\nabla\psi|^{2}+f^{2})}{R^{2}}=\beta(\psi)+\frac{\alpha(\psi)+f^{2}(\psi)}{R^{2}}. \tag{5}$$

III. CONSTRUCTION OF DEGENERATE EQUILIBRIA

In this section we seek solutions of Eq. (2) that have the "degeneracy" property (4). This is an unusual problem. As we have already indicated one usually solves Eq. (2) in a given boundary for known functions $p(\psi)$ and $f(\psi)$. In the present problem the boundary and the functions $p(\psi)$ and $f(\psi)$, as well as $\alpha(\psi)$ and $\beta(\psi)$, all have to be determined.

We write

$$\nabla\psi=R^{2}g(\psi,R)\mathbf{e}, \tag{6}$$

where $\mathbf{e}$ is a unit vector $(e_R,0,e_Z)$ and $g^2=[\alpha(\psi)+\beta(\psi)R^2]/R^4$. Then using the relation

$$\nabla\cdot\mathbf{e}=\frac{1}{R}\frac{\partial}{\partial R}(Re_R)\Big|_{\psi}, \tag{7}$$

the Grad–Shafranov equation (2) can be reduced to

$$\frac{\partial}{\partial R}(Rge_R)\Big|_{\psi}=RL(\psi)+\frac{1}{R}M(\psi), \tag{8}$$

where

$$\begin{aligned}L(\psi)&\equiv-(\mu_0 p'+\beta'/2),\\ M(\psi)&\equiv-(ff'+\alpha'/2).\end{aligned} \tag{9}$$

Therefore

$$ge_R=\frac{R}{2}L(\psi)+\frac{\log R}{R}M(\psi)+\frac{1}{R}C(\psi), \tag{10}$$

which defines a single surface in terms of the three parameters L,M,C. If we introduce R_0 as the radius at which e_R

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$=0$ (i.e., where the surface is tangential to the R axis), Eq. (10) becomes

$$e_R = h(\psi,R)/g(\psi,R), \quad e_z = (1 - h^2/g^2)^{1/2}, \tag{11}$$

with

$$h(\psi,R) = \frac{L(\psi)}{2}\frac{(R^2 - R_0^2)}{R} + \frac{M(\psi)}{R}\log\frac{R}{R_0} \tag{12}$$

and

$$g(\psi,R) = [\alpha(\psi) + \beta(\psi)R^2]^{1/2}/R^2. \tag{13}$$

Several conditions have yet to be imposed if the single surface defined by (10) or (11) is to be part of a global equilibrium. One of these is that the surface should be a smooth closed curve—but it is convenient to defer consideration of this until other conditions have been dealt with. These arise from the requirement that

$$\nabla\times(R^2\mathbf{g}\mathbf{e}) = 0, \tag{14}$$

which relates adjacent surfaces. Using (11) for e_R and e_Z, introducing (R,ψ) as independent variables and carrying out some manipulation, (14) leads to the condition

$$\frac{R^3h}{2}\frac{\partial g^2}{\partial\psi} - R^3g^2\frac{\partial h}{\partial\psi} + \frac{R}{2}\frac{\partial g^2}{\partial R} - \frac{R}{2}\frac{\partial h^2}{\partial R} + 2(g^2 - h^2) = 0. \tag{15}$$

At this point it is convenient to introduce new quantities

$$P \equiv \alpha/L^2, \quad Q \equiv \beta/L^2, \quad \text{and } X \equiv R_0^2. \tag{16}$$

Then, when g and h are introduced into Eq. (15), it is reduced to a polynomial in R and $\log R$ with coefficients that are functions of ψ. To satisfy this for all values of ψ and R the coefficients must vanish so that Eq. (15) can be satisfied if, and only if, four conditions are met. The first of these is that $M \equiv 0$. This means that $\alpha(\psi)$ (a property of the solution to the Grad–Shafranov equation) is related to $f(\psi)$ (part of the specification of the equation) by $\alpha + f^2 = \text{const}$. Hence in degenerate equilibria the total magnetic field must be of the form

$$B^2 = \beta(\psi) + k/R^2.$$

Such equilibria have been investigated by Palumbo.[2] If $k=0$, they are "isodynamic," that is, the flux surfaces are also surfaces of constant B^2. The remaining three conditions implied by (15) are in the form of ordinary differential equations that must be satisfied by P,Q,X. They can be written

$$\text{(i)}\quad \frac{dP}{d\lambda} = \frac{4Q^2}{P+XQ} - X,$$
$$\text{(ii)}\quad \frac{dQ}{d\lambda} = 3, \tag{17}$$
$$\text{(iii)}\quad \frac{dX}{d\lambda} = \frac{-2Q}{P+XQ},$$

where $d\lambda = d\psi/L(\psi)$.

We must now consider the surface closure condition mentioned earlier. When $M=0$, the equation for a magnetic surface obtained from (11)–(13) becomes

$$\frac{dZ}{dx} = \frac{-(x-X)}{4[P+Qx-x(x-X)^2/4]^{1/2}}, \tag{18}$$

where $x = R^2$. Then the closure of the surfaces provides a further relation between P, Q, and X,

$$F(P/X^3,Q/X^2) \equiv \int \frac{(s-1)ds}{[P/X^3 + sQ/X^2 - s(s-1)^2/4]^{1/2}} = 0, \tag{19}$$

where the integration is between the two largest roots of the denominator. The integral can be expressed in terms of complete elliptic integrals.

Note that $L(\psi)$ no longer appears explicitly in the problem and can be chosen arbitrarily. This reflects the degeneracy of the plasma equilibria we are seeking. Also all the equations are invariant under a scale transformation $X\to\mu X$, $P\to\mu^3P$, $Q\to\mu^2Q$, $\lambda\to\mu^2\lambda$, which merely magnifies and relabels the surfaces.

The magnetic surfaces of the degenerate equilibria can now be constructed. We use the scale invariance to set $X=1$ at the magnetic axis. Then from the closure condition Eq. (19) we find (see the Appendix) that near the magnetic axis $P\to 0$, $Q\to 0$, with $P/Q = -1/3$. Using these as starting values, $P(\lambda)$, $Q(\lambda)$, and $X(\lambda)$ are computed from the differential equations (17). We choose the origin of λ so that $\lambda = 0$ on axis. Then $Q(\lambda) = 3\lambda$ and the computed values for $P(\lambda)$ and $X(\lambda)$ are shown in Figs. 1 and 2. [A very good approximation is $P(\lambda) \approx \lambda(10\lambda - 1)$ and $X(\lambda) \approx 1 - 3\lambda$.] Once $P(\lambda)$, $Q(\lambda)$, and $X(\lambda)$ are found the individual magnetic surfaces are computed from (18). In this way the complete configuration is built up surface by surface. The result is shown in Fig. 3, which agrees with Palumbo's calculation.[3] It is an important feature of this construction (see the Appendix) that if the initial surface is closed, so are all others. It is also clear that the surfaces constructed in this way are unique apart from the scale factor already noted.

At this point it is useful to summarize the construction of the degenerate magnetic configuration. We first obtain an equation satisfied by a *single* magnetic surface; this contains three unknown parameters. The condition that this single surface be part of an equilibrium imposes constraints on these three parameters in the form of differential equations and the condition that the surfaces be closed provides the initial values for these differential equations. Their solution then provides the value of the three parameters on each magnetic surface. Note that the equilibria are obtained without at any time needing to solve a partial differential equation.

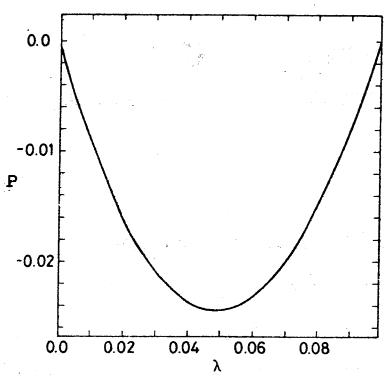

FIG. 1. The function $P(\lambda)$.

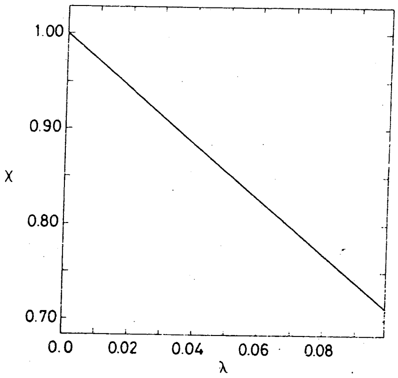

FIG. 2. The function $X(\lambda)$.

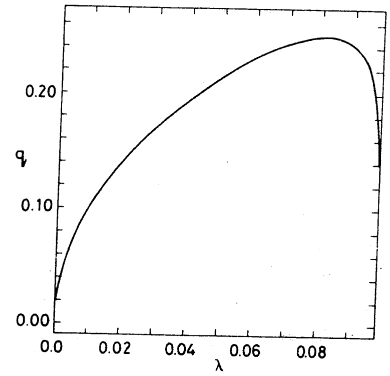

FIG. 4. The function $q(\lambda)$.

IV. PROPERTIES OF THE EQUILIBRIA

In this section we review some of the properties of the degenerate equilibria associated with the surfaces of Fig. 3. The unknown functions α, β, p, and f, which describe the equilibria, are functions of $L(\lambda)$. Thus

$$\alpha(\lambda) = L^2(\lambda)P(\lambda),$$

$$\beta(\lambda) = L^2(\lambda)Q(\lambda),$$

$$\mu_0 \frac{\partial}{\partial\lambda} p(\lambda) = -L^2(\lambda) - \frac{\partial}{\partial\lambda}\left(\frac{L^2Q}{2}\right),$$

$$\frac{\partial}{\partial\lambda}(f^2 + \alpha) = 0. \tag{20}$$

The equilibria therefore involve two constants of integration and the arbitrary function $L(\lambda)$. The following features depend on $L(\lambda)$. The toroidal field

$$B_\phi^2 = -L^2(\lambda)P(\lambda)/R^2 + k/R^2. \tag{21}$$

The poloidal field

$$B_p^2 = L^2(\lambda)[Q(\lambda) + P(\lambda)/R^2]. \tag{22}$$

The total field

$$B^2 = L^2(\lambda)Q(\lambda) + k/R^2. \tag{23}$$

The toroidal current density

$$j_\phi = -R\left(L + \frac{1}{2L}\frac{d}{d\lambda}(L^2Q)\right) - \frac{1}{2LR}\frac{d}{d\lambda}(L^2P) \tag{24}$$

and the plasma pressure

$$p(\lambda) = p_0 - \frac{L^2Q}{2\mu_0} - \frac{1}{\mu_0}\int_0^\lambda L^2(\lambda')d\lambda'. \tag{25}$$

The integration constant p_0 can be used to set the pressure to zero on any chosen surface. If the other constant of integration k is set equal to zero we have the isodynamic equilibria introduced by Palumbo,[2] in which constant-B surfaces and flux surfaces coincide. Furthermore, the field strength B^2 increases with distance from the magnetic axis if $\lambda L^2(\lambda)$ is an increasing function of λ so that minimum-B equilibria are included. [However, $(p + B^2/2)$ is always a decreasing function of λ, and B^2 vanishes on the magnetic axis so that minimum-B fields exist only for finite β.] Some features of isodynamic equilibria are independent of L and of the pres-

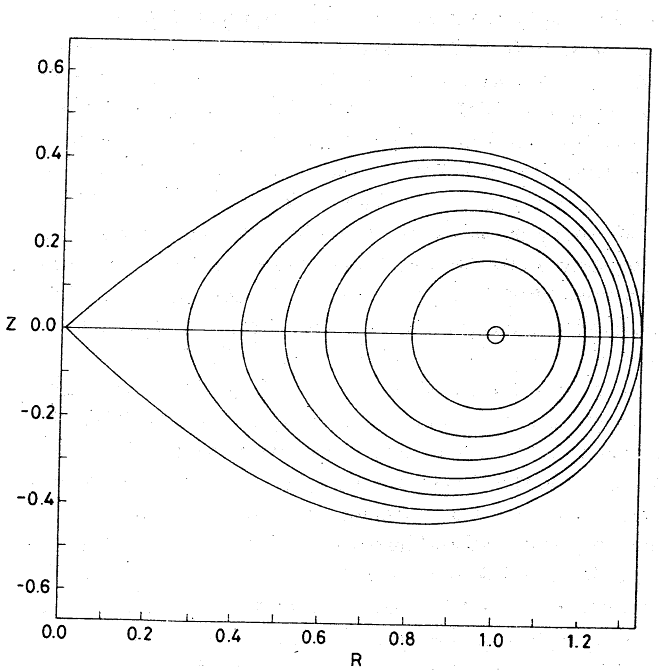

FIG. 3. The unique magnetic surfaces of degenerate equilibria.

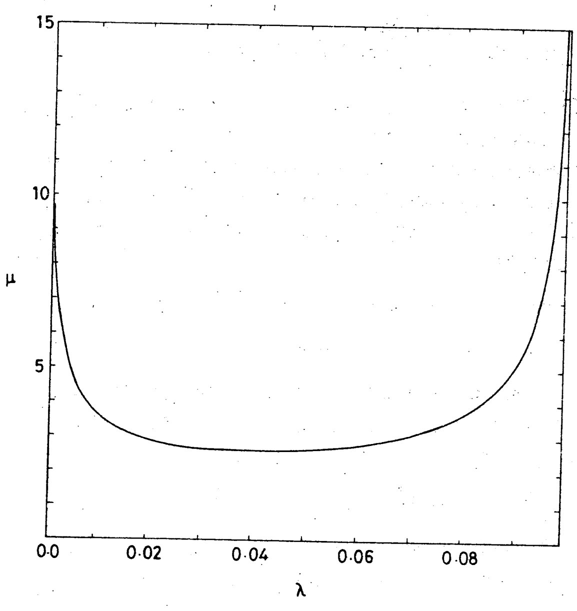

FIG. 5. The function $\mu(\lambda)$.

sure profile. They include the safety factor q, shown in Fig. 4, and $\mu \equiv |\mathbf{j}\cdot\mathbf{B}|/B^2$,

$$\mu = -\frac{d}{d\lambda}|P|^{1/2} + \frac{5}{6\lambda}|P|^{1/2}, \qquad (26)$$

which is shown in Fig. 5. It will be noted that in the isodynamic equilibria both the poloidal and toroidal magnetic fields vanish on axis, but in such a way that q is also zero on axis. On the other hand μ is singular on axis, but is nevertheless substantially constant across a large part of the available aperture.

V. STABILITY OF MINIMUM-B EQUILIBRIA

Minimum-B *mirror* systems with a pressure tensor $p_\perp = p_\perp(B)$, $p_\parallel = p_\parallel(B)$ are almost unconditionally stable.[4] However, Palumbo[2] has pointed out that isodynamic toroidal equilibria are always unstable according to the Mercier criterion,[5] at least near the magnetic axis. Hence the toroidal minimum-B equilibria mentioned above are also unstable, despite the fact that they have the analogous property $p = p(B)$. It is a worthwhile examining how this distinction arises.

The energy integral for anisotropic plasma can be expressed in the form[6]

$$\delta W = \mathbf{Q}_\perp^2[1 + (p_\perp - p_\parallel)/B^2] + Q_\parallel^2[1 + (2p_\perp + C)/B^2] - j_\parallel \mathbf{n}\cdot\mathbf{Q}_\perp \times \boldsymbol{\xi}[1 + (p_\perp - p_\parallel)/B^2]$$
$$+ q[\boldsymbol{\xi}\cdot\nabla p_\parallel + (p_\perp - p_\parallel)s] - (2Q_\parallel/B + s)[\boldsymbol{\xi}\cdot\nabla p_\perp - (2p_\perp + C)s] + \kappa^2, \qquad (27)$$

where $s = (\boldsymbol{\xi}\cdot\nabla B)/B$, $q = -\boldsymbol{\xi}\cdot(\mathbf{n}\cdot\nabla)\mathbf{n}$, and $\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B})$. The last term in δW is a contribution of the trapped particles and C is a moment of the distribution function. We also have the equilibrium relations

$$\frac{\partial p_\parallel}{\partial s} = \frac{(p_\parallel - p_\perp)}{B}\frac{\partial B}{\partial s},$$
$$\frac{\partial p_\perp}{\partial s} = \frac{(C + 2p_\perp)}{B}\frac{\partial B}{\partial s}. \qquad (28)$$

Consequently, for mirror equilibria with $p_\perp = p_\perp(B)$, $p_\parallel = p_\parallel(B)$, where the parallel current is zero, the energy integral reduces to

$$\delta W = \mathbf{Q}_\perp^2[1 + (p_\perp - p_\parallel)/B^2] + Q_\parallel^2[1 + (2p_\perp + C)/B^2] + \kappa^2. \qquad (29)$$

This is automatically positive for all disturbances provided only that the weak requirements (for mirror and firehose stability)

$$B^2 + p_\perp > p_\parallel, \quad \frac{dp_\perp(B)}{dB} + B > 0 \qquad (30)$$

are satisfied. Therefore these mirror minimum-B equilibria are always stable, even at finite β.

On the other hand, the corresponding energy principle for scalar-pressure plasma [obtained from (27) by setting $p_\perp = p_\parallel$, $(2C + p_\perp) = 0$, and dropping the trapped particle contribution] is

$$\delta W = \mathbf{Q}_\perp^2 + Q_\parallel^2 - j_\parallel(\mathbf{n}\cdot\mathbf{Q}_\perp\times\boldsymbol{\xi}) + \boldsymbol{\xi}\cdot\nabla p(q - s - 2Q_\parallel/B). \qquad (31)$$

This has to be minimized over the two-component vector $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp$, but it is permissible to carry out one of the minimizations over $Q_\parallel$. Then

$$\delta W = \mathbf{Q}_\perp^2 - j_\parallel(\mathbf{n}\cdot\mathbf{Q}_\perp\times\boldsymbol{\xi}) + 2(\boldsymbol{\xi}\cdot\nabla p)[\boldsymbol{\xi}\cdot\nabla(p + B^2/2)], \qquad (32)$$

where we have introduced the explicit form for s and expressed q as $-\boldsymbol{\epsilon}\cdot\nabla(p + B^2/2)B^2$.

It can be seen from this that even though $p = p(B)$ potential sources of instability associated with both ∇p and with $j_\parallel$ remain. Consequently, unlike the mirror case, no general conclusion about stability follows from the fact that $p = p(B)$ in toroidal, scalar pressure, systems. One reason (in addition to the presence of $j_\parallel$) why the stability of mirror equilibria does not carry over to scalar pressure can be seen from Eq. (28). With $p_\parallel = p_\parallel(B)$ this implies

$$\frac{dp_\parallel(B)}{dB} = \frac{p_\perp - p_\parallel}{B},$$

and the scalar pressure limit of this is $p = \text{const}$!

VI. CONCLUSIONS

We have shown that there is a unique configuration of toroidal magnetic surfaces that corresponds to more than one equilibrium. In this respect it is the analog of a set of concentric circular surfaces for the cylindrical equilibria. It is also the configuration of the isodynamic equilibria. We constructed this degenerate configuration (Fig. 3) by a novel approach to the Grad–Shafranov equation in which the magnetic surfaces are calculated individually. Each surface is defined by an ordinary differential equation containing three parameters. These parameters satisfy differential equations representing the fact that each surface is part of a global equilibrium and the initial values for these equations are determined by the requirement that the surfaces are closed.

We have also noted that among the isodynamic equilibria there are toroidal minimum-B equilibria in which B increases and p decreases everywhere with the distance from the magnetic axis. (They are possible despite the well-known result[7] that there are no similar vacuum fields because they exist only at high-β and have no low-β limit.) Such equilibria differ from tokamaks in that $B_p \gtrsim B_\phi$ and from the toroidal pinch in that $q(\psi)$ is increasing with minor radius rather than decreasing, and from both in that B and q are zero on the magnetic axis. Despite their minimum-B character, the fact that $p = p(B)$ and that all guiding center drifts lie in the magnetic surface (i.e., the equilibria are omnigeneous[8,9]), they do not possess the intrinsic stability of their mirror counterparts. Nevertheless, their freedom from neoclassical and trapped particle effects might make them interesting if examples with gross stability could be found. In this respect, an instability to localized modes near the axis need not be catastrophic—any more than it is in tokamaks with $q < 1$.

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APPENDIX A: THE SURFACE CLOSURE CONDITION

A magnetic surface is described by the equation

$$\frac{dZ}{dx} = \frac{-(x-X)}{4[P+Qx-x(x-X)^2/4]^{1/2}}, \tag{A1}$$

which contains three parameters $P(\psi)$, $Q(\psi)$, and $X(\psi)$. These parameters themselves must satisfy the differential equations (17). However, in addition, one must ensure that the surface generated by (A1) is smooth and closed. This requires that the three roots s_0, s_1, s_2 of the denominator be real and

$$\int_{s_1}^{1} + \int_{1}^{s_2} \frac{(s-1)ds}{[P/X^2 + sQ/X^3 - s(s-1)^2/4]^{1/2}} = 0, \tag{A2}$$

where s_1 and s_2 are the two largest roots. [The curve is parallel to the Z axis at $s = s_1$ and $s = s_2$ and parallel to the R axis at $s = 1$.] This closure condition can be expressed in terms of complete elliptic integrals of the first and second kind as

$$E(a) + abK(b) = 0, \tag{A3}$$

where $a = (s_2 - s_1)/(s_2 - s_0)$ and $b = (s_0 - 1)/(s_2 - s_1)$.

At the magnetic axis s_1 and s_2 are coincident and the roots become 0,1,1. This occurs when $P/X^3 \to 0$, $Q/X^2 \to 0$. In the neighborhood of the axis P/X^3 and Q/X^2 are small and the roots are

$$s_0 \simeq \frac{4P}{X^3}, \quad s_1, s_2 = 1 - \frac{2P}{X^3} \pm 2\left(\frac{P}{X^3} + \frac{Q}{X^2}\right)^{1/2}. \tag{A4}$$

In this limit the closure condition (A2) or (A3) can readily be evaluated to show that in the vicinity of the axis

$$Q/X^2 = -3P/X^3. \tag{A5}$$

It can be verified that this is compatible, in the appropriate limit, with the differential equations (17). We have, in fact, verified that the *full* closure condition (A3) is consistent with these differential equations. This means that if one ensures that the initial values of P,Q,X correspond to a closed surface then closure is ensured for all other surfaces. Closure, although crucial to the calculation, is required only to give initial values for the differential equations (17); thereafter it follows automatically.

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